Axisymmetric Plasma Flows in External Magnetic Fields with Hall Effect *

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The present analysis refers to steady, barotropic, and inviscid flows of rarefied gas plasmas, both as free jet and within a cylindrical channel, with superimposed electric current, under the influence of an axisymmetric external magnetic field. The non-dimensionalized equations of magnetogasdynamics are linearized by expanding the unknown functions in double power series in terms of the magnetic force number and the magnetic Reynolds number, both assumed smaller than one. For Hall parameters sufficiently small compared to one and slightly curved magnetic lines of force, closed form solutions for the electric current density, as well as plasma velocity and mass density are derived and discussed. Increases in the axial velocity component are achieved more efficiently by using free jet rather than channel flow.

1. Introduction

The study of axisymmetric plasma flow — both as free jet and within a channel — under the influence of an appropriately chosen external (or imposed) magnetic field is of particular interest for developing magnetoplasmadynamic (MPD) accelerators and propulsion devices. This subject has received considerable attention by several authors [1–5] who investigated various aspects of it with similar assumptions and methods as done in the present paper. Here, we aim at developing — within the framework of continuum theory — an analytic procedure to yield three-dimensional (axisymmetric) closed form solutions, allowing for Hall effect as well as for an axially superimposed electric current. For this, we use a very simple expression for the imposed magnetic induction and a classical linearization method as already employed by different authors [6–8]. Dimensional magnitudes are marked by a roof ^.

2. Definition of the Problem

The steady, inviscid, and barotropic flow of a diffuse arc plasma between two equipotential planes, a distance $\hat{L}$ apart, under the influence of an imposed axisymmetric magnetic induction $\hat{\boldsymbol{B}}^{00}$ is to be analyzed, taking into account the Hall effect which is assumed to be small. Free jet and channel flow are dealt with together, $\hat{R}$ being the radius of the unperturbed free jet or of the circular tube, respectively (Fig. 1).

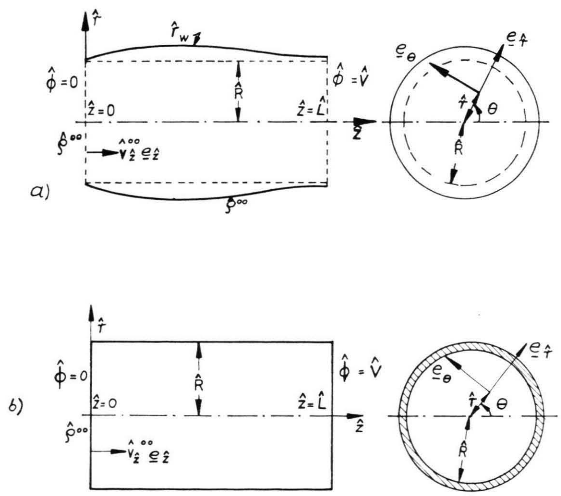

Fig. 1. Geometric configuration for free jet (a) and channel (b) flow.

Besides the potential difference $\hat{V}$, there are also given the initial velocity $\hat{\boldsymbol{v}}^{00}\{0, 0, \hat{v}_{\hat{z}}^{00}\}$, $\hat{v}_{\hat{z}}^{00} = \text{const}$, and the constant mass density $\hat{\varrho}^{00}$ at the entrance section $\hat{z} = 0$ (Fig. 1). For free jet flow, $\hat{\varrho}^{00}$ is assumed to equal the ambient neutral gas density. Since there shall not exist an impressed pressure gradient, all acceleration effects are due only to the Lorentz-force. Our main objective is to determine the electric current density $\hat{\boldsymbol{j}}$ as well as the plasma velocity $\hat{\boldsymbol{v}}$ and its mass density $\hat{\varrho}$ within the bounded

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region $0 \leqq \hat{r} \leqq \hat{R}$, $0 \leqq \hat{z} \leqq \hat{L}$. For free jet flow, the deformed jet boundary $\hat{r}_w$ is also required (Fig. 1a).

All material coefficients are assumed constant. In the generalized Ohm's law, ion-slip and electron gas partial pressure gradient are being neglected.

3. Fundamental Equations

The magnetogasdynamic equations (referred to a fixed frame) — for the MKSA-system of units and in one-fluid description — which govern the steady and barotropic flow of an inviscid plasma read as follows[9,10]:

$$\hat{\nabla} \cdot \hat{\varrho} \hat{\boldsymbol{v}} = 0 , \tag{1}$$

$$\hat{\varrho} (\hat{\boldsymbol{v}} \cdot \hat{\nabla}) \hat{\boldsymbol{v}} = - \hat{\nabla} \hat{\varrho} (\mathrm{d}\hat{p}/\mathrm{d}\hat{\varrho}) + \hat{\boldsymbol{j}} \times \hat{\boldsymbol{B}} = - \hat{a}^2 \hat{\nabla} \hat{\varrho} + \hat{\boldsymbol{j}} \times \hat{\boldsymbol{B}} , \tag{2}$$

$$\hat{\nabla} \times \hat{\boldsymbol{B}}^i / \hat{\mu} = \hat{\boldsymbol{j}} , \tag{3}$$

$$\hat{\nabla} \cdot \hat{\boldsymbol{B}} = 0 , \tag{4}$$

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{B}}^{00} + \hat{\boldsymbol{B}}^i , \tag{5}$$

$$\hat{\nabla} \times \hat{\boldsymbol{E}} = 0 , \tag{6}$$

$$\hat{\boldsymbol{j}} = \hat{\sigma} (\hat{\boldsymbol{E}} + \hat{\boldsymbol{v}} \times \hat{\boldsymbol{B}}) - (\psi / \hat{B}_0) (\hat{\boldsymbol{j}} \times \hat{\boldsymbol{B}}) . \tag{7}$$

Herein are:

$\hat{p} = \hat{p}(\hat{\varrho})$	gas pressure,
$\hat{a}$	speed of sound,
$\hat{\boldsymbol{B}}$	magnetic induction,
$\hat{\boldsymbol{B}}^{00}$, $\hat{\boldsymbol{B}}^{\mathrm{i}}$	imposed and induced parts thereof, respectively,
$\hat{\mu}$	magnetic permeability,
$\hat{\boldsymbol{E}}$	electric field,
$\hat{\sigma}$	scalar electrical conductivity,
ψ	Hall parameter,
$\hat{B}_0 \equiv \lvert \hat{\boldsymbol{B}}^{00}(0, \theta, 0) \rvert$	reference value of the magnetic induction.

In order to non-dimensionalize the fundamental equations (1)—(7) we introduce, in addition to $\hat{B}_0$, the following reference values taken at the entrance section $\hat{z} = 0$: for lengths the radius $\hat{R}$, for velocities and mass density the given values $\hat{v}_{\hat{z}}^{00}$ and $\hat{\varrho}^{00}$, respectively, for the electric field the expression $\hat{E}_0 \equiv \hat{v}_{\hat{z}}^{00} \hat{B}_0$, and $\hat{\sigma}_0 \hat{E}_0$ for the electric current density.

The fundamental equations in non-dimensional form then read:

$$\nabla \cdot \varrho \boldsymbol{v} = 0 , \tag{8}$$

$$\varrho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = - (1/M^2) \nabla \varrho + S R_{\mathrm{m}} (\boldsymbol{j} \times \boldsymbol{B}) , \tag{9}$$

$$\nabla \times \boldsymbol{B}^i = R_{\mathrm{m}} \boldsymbol{j} , \tag{10}$$

$$\nabla \cdot \boldsymbol{B} = 0 , \tag{11}$$

$$\boldsymbol{B} = \boldsymbol{B}^{00} + \boldsymbol{B}^i , \tag{12}$$

$$\nabla \times \boldsymbol{E} = 0 , \tag{13}$$

$$\boldsymbol{j} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \psi (\boldsymbol{j} \times \boldsymbol{B}) . \tag{14}$$

In addition to the Hall parameter ψ the following dimensionless ratios appear herein:

free stream Mach number: $M = \hat{v}_{\hat{z}}^{00} / \hat{a} = 1/a$, (15)

magnetic force number: $S = \hat{B}_0{}^2 / \hat{\mu} \, \hat{\varrho}^{00} \, \hat{v}_{\hat{z}}^{00 2}$, (16)

magnetic Reynolds number: $R_{\mathrm{m}} = \hat{\mu} \, \hat{\sigma} \, \hat{R} \, \hat{v}_{\hat{z}}^{00}$. (17)

Next, we transform Ohm's law (14) by eliminating[10] $\boldsymbol{j}$ on the right-hand side of (14) and expressing — because of Eq. (13) — the electric field $\boldsymbol{E}$ in terms of the electric potential Φ as $\boldsymbol{E} = - \nabla \Phi$; this yields:

$$\boldsymbol{j} (1 + \psi^2 B^2) = - \nabla \Phi + \boldsymbol{v} \times \boldsymbol{B} + \psi (\nabla \Phi - \boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{B} - \psi^2 (\nabla \Phi \cdot \boldsymbol{B}) \boldsymbol{B} . \tag{18}$$

The 14 unknown variables ϱ, $\boldsymbol{v}$, $\boldsymbol{B}^i$, $\boldsymbol{B}$, $\boldsymbol{j}$, and Φ are uniquely defined by 14 equations: (8)—(12) and (18) — counting componentwise.

The imposed magnetic induction shall be given as follows:

$$\boldsymbol{B}^{00} = \{ r / 2 z_0 , \, 0, \, (z_0 - z) / z_0 \} . \tag{19}$$

This expression obviously represents — as required — an axisymmetric, solenoidal, and irrotational field. Furthermore, it proves to be well suited for dealing with analytical problems. Depending on the choice of the dimensionless constant z_0, a variety of magnetic field configurations may be simulated within a properly chosen bounded region[10]. In the following, it will be assumed that

$$|z_0| \gg L > 1 , \quad L = \hat{L} / \hat{R} . \tag{20}$$

Within the bounded region of interest $0 \leqq r \leqq 1$, $0 \leqq z \leqq L$, the imposed induction $\boldsymbol{B}^{00}$ will then be represented by a field of only slightly curved lines of force, similar to the one originated by a half-coil. This field diverges or converges in the (streamwise) positive z-direction for positive or negative values of z_0, respectively. For $z_0 = \pm \infty$, it degenerates into the homogeneous, purely axial field of the semi-infinite coil.

The above introduced parameters are supposed to satisfy the following inequalities:

$$S \ll 1 , \quad R_{\mathrm{m}} \ll 1 , \tag{21}$$

$$\psi^2 \ll 1 . \tag{22}$$

No assumptions regarding M are necessary at this time.

Because of (19), (20), and (21) the following order of magnitude estimate holds:

$$0(|\boldsymbol{B}|) = 0(|\boldsymbol{B}^{00}|) \lesssim 1\,. \tag{23}$$

Therefore and on account of (22), the term $\psi^2 B^2$ on the left-hand side of Eq. (18) may be neglected relative to one.

4. Solution of the Problem

4.1. *Linearization of the Fundamental Equations*

In order to linearize the non-dimensional fundamental equations given in 3., the unknown variables are expanded in double power series in terms of the parameters R_{m} and S what is symbolically represented by:

$$U = U^{00} + R_{\mathrm{m}}\, U^{01} + S\, U^{10} + R_{\mathrm{m}}^2\, U^{001} + R_{\mathrm{m}} S\, U^{010} + S^2\, U^{100} + \cdots,$$

with:

$$U = \varrho,\ \boldsymbol{v},\ \boldsymbol{B},\ \boldsymbol{j},\ \Phi\,. \tag{24}$$

Substituting the expansions (24) into the fundamental equations, and subsequently equating the coefficients of like powers of R_{m} and S, one obtains linear equations for determining the expansion coefficients or "perturbations" [10].

The purely gasdynamic solution for the unperturbed ground state (i.e. without MPD-interactions, $R_{\mathrm{m}} = S = 0$) is given by:

$$\boldsymbol{v} = \boldsymbol{v}^{00} = \{0, 0, 1\}\,, \qquad \varrho = \varrho^{00} = 1\,. \tag{25}$$

Perturbed solutions differing therefrom are obtained only if the Lorentz-force term in the momentum Eq. (9) is taken into consideration. This requires considering at least terms up to the order $R_{\mathrm{m}} S$ in (24). Such (first-order) perturbation analysis in $\boldsymbol{v}$ and ϱ then yields the following:

$$\boldsymbol{v}^{01} \equiv 0 \equiv \boldsymbol{v}^{10}\,, \quad \varrho^{01} \equiv 0 \equiv \varrho^{10}\,, \quad \boldsymbol{v}^{001} \equiv 0 \equiv \boldsymbol{v}^{100}\,, \quad \varrho^{001} \equiv 0 \equiv \varrho^{100}\,, \tag{26}$$

$$\nabla \cdot (\boldsymbol{v}^{010} + \varrho^{010}\, \boldsymbol{v}^{00}) = 0\,, \tag{27}$$

$$(\boldsymbol{v}^{00} \cdot \nabla)\, \boldsymbol{v}^{010} + (1/M^2)\, \nabla \varrho^{010} = \boldsymbol{j}^{00} \times \boldsymbol{B}^{00}\,. \tag{28}$$

Thus, according to Eqs. (27) and (28), only the imposed magnetic induction $\boldsymbol{B}^{00}$ and the ground state value of the electric current density, $\boldsymbol{j}^{00}$, are required for determining the first-order perturbations $\boldsymbol{v}^{010}$ and ϱ^{010}. $\boldsymbol{j}^{00}$ follows from Ohm's law for the ground state obtained by inserting (22) and (24) into (18):

$$\boldsymbol{j}^{00} = -\nabla \Phi^{00} + \boldsymbol{v}^{00} \times \boldsymbol{B}^{00} + \psi(\nabla \Phi^{00} - \boldsymbol{v}^{00} \times \boldsymbol{B}^{00}) \times \boldsymbol{B}^{00} - \psi^2 (\nabla \Phi^{00} \cdot \boldsymbol{B}^{00})\, \boldsymbol{B}^{00}\,. \tag{29}$$

4.2. *Electric Current Density*

First of all, the ground state electric potential, Φ^{00}, is calculated by substituting (29) into the equation

$$\nabla \cdot \boldsymbol{j}^{00} = 0 \tag{30}$$

which is a consequence of (10) and (24). Taking into account that $\boldsymbol{B}^{00}$ is irrotational, we obtain the following elliptic-type differential equation for Φ^{00} [10]:

$$\Delta \Phi^{00} + \psi^2\, \boldsymbol{B}^{00} \cdot [\nabla (\nabla \Phi^{00} \cdot \boldsymbol{B}^{00})] = \frac{\psi}{2} \frac{\partial}{\partial z} (\boldsymbol{B}^{00})^2\,. \tag{31}$$

Substituting herein $\boldsymbol{B}^{00}$ by its value (19) and neglecting terms of the order ψ^2 or smaller in relation to unity, Eq. (31) yields:

$$\Delta \Phi^{00} = -(\psi/z_0^2)(z_0 - z)\,[1 + \psi(r\, \Phi^{00}_{|rz} - \Phi^{00}_{|z})]\,. \tag{32}$$

[Here and in the following we use the abbreviation $W_{|xy}$ for $\partial^2 W/\partial x\, \partial y$.]

The existence of two equipotential planes at $z = 0$ and $z = L$, respectively, is being expressed, without loss of generality, by the following boundary conditions:

$$\Phi^{00}(r, \theta, 0) = 0, \quad \Phi^{00}(r, \theta, L) = V; \quad 0 \leqq r \leqq 1\,, \tag{33}$$

with $V = \hat{V}/(\hat{E}_0\, \hat{R})$.

For physical reasons, the radial component of the electric current density, j_r^{00}, must vanish on the axis $r = 0$ and for $r = 1$, i.e. on the undeformed free jet boundary or at the channel wall, respectively. This leads, with (19), (22), (25), and (29) to the following remaining boundary conditions:

$$\Phi^{00}_{|r}(0, \theta, z) = 0\,, \quad \Phi^{00}_{|r}(1, \theta, z) = -\psi \frac{z_0 - z}{2 z_0^2}\,[1 + \psi\, \Phi^{00}_{|z}(1, \theta, z)]\,, \quad 0 \leqq z \leqq L\,. \tag{34}$$

The derivatives of the dependent variable Φ^{00} on the right-hand side of (32) and $(34)_2$ are multiplied by the Hall parameter ψ which in (22) was assumed much smaller than one. These derivatives may thus be represented as follows:

$$\Phi^{00}_{|z} = V/L + 0(\psi)\,; \quad \Phi^{00}_{|rz} = 0(\psi)\,, \tag{35}$$

which may be confirmed afterwards. Since terms of the order ψ^2 will be neglected relative to one, the accuracy requirements adopted here suggest substituting in (32) and $(34)_2$ $\Phi^{00}_{|z}$ by V/L and to omit $\Phi^{00}_{|rz}$ altogether. For the resulting Poisson's equation for Φ^{00} we seek solutions of the form

$$\Phi^{00}(r,z) = \Phi_1(r,z) + \Phi_2(r,z)\,. \qquad (36)$$

(The azimuthal angle θ is omitted now since we are dealing with an axisymmetric problem.) Φ_1 and Φ_2 are defined to satisfy the following boundary value problems:

$$\Delta\Phi_1 = (\psi/z_0{}^2)\,(1-\psi\,V/L)\,(z-z_0)\,, \qquad (37)$$

$$\Phi_1(r,0) = 0\,, \quad \Phi_1(r,L) = 0\,, \qquad (38)$$

$$\Phi_{1|r}(0,z) = 0\,, \qquad (39)$$

$$\Phi_{1|r}(1,z) = (\psi/2\,z_0^2)\,(1+\psi\,V/L)\,(z-z_0)\,;$$

and

$$\Delta\Phi_2 = 0\,, \qquad (40)$$

$$\Phi_2(r,0) = 0\,, \quad \Phi_2(r,L) = V\,, \qquad (41)$$

$$\Phi_{2|r}(0,z) = 0\,, \quad \Phi_{2|r}(1,z) = 0\,. \qquad (42)$$

In order to solve the boundary value problem for Φ_1, the right-hand side of Eq. (37) is expanded in terms of the eigenfunctions $\sin \nu\pi z/L$, $\nu = 1, 2, \ldots$, to yield:

$$\frac{\psi}{z_0{}^2}\left(1-\psi\frac{V}{L}\right)(z-z_0) = \frac{2\psi}{\pi\,z_0{}^2}\left(1-\psi\frac{V}{L}\right)\sum_{\mu}\frac{P_\mu}{\mu}\sin\frac{\mu\,\pi}{L}z\,, \qquad (43)$$

wherein:

$$P_\mu \equiv (-1)^{\mu+1}L + [(-1)^\mu - 1]\,z_0\,, \quad \mu = 1, 2, \ldots. \qquad (44)$$

Assuming a solution $\Phi_1(r,z)$ exists, we now attempt to determine its Fourier coefficients

$$\varphi_\nu(r) = \frac{2}{L}\int_0^L \Phi_1(r,z)\sin\frac{\nu\,\pi}{L}z\,\mathrm{d}z \qquad (45)$$

in terms of the orthogonal and complete system of functions $\sin\nu\pi z/L$, $\nu = 1, 2, \ldots$. To this end, we multiply Eq. (37) by the eigenfunction $\sin\nu\pi z/L$ and integrate termwise over the fundamental domain which gives:

$$\int_0^L\left(\Phi_{1|rr} + \frac{1}{r}\Phi_{1|r} + \Phi_{1|zz}\right)\sin\frac{\nu\,\pi}{L}z\,\mathrm{d}z = \frac{\psi L}{\pi\,z_0{}^2}\left(1-\psi\frac{V}{L}\right)\frac{P_\nu}{\nu}\,. \qquad (46)$$

Integrating by parts and using (45) and (38), one obtains the following inhomogeneous Bessel's equation for the Fourier coefficients[10]:

$$\varphi_\nu''(r) + \frac{1}{r}\varphi_\nu'(r) - \left(\frac{\nu\,\pi}{L}\right)^2\varphi_\nu(r) = \frac{2\psi}{\pi\,z_0{}^2}\left(1-\psi\frac{V}{L}\right)\frac{P_\nu}{\nu}\,, \quad \nu = 1, 2, \ldots. \qquad (47)$$

Inverting relation (45) and using the Fourier coefficients $\varphi_\nu(r)$ — to be determined from Eq. (47) — leads to the following solution of Poisson's Eq. (37) fulfilling the boundary conditions (39):

$$\Phi_1(r,z) = \frac{\psi L}{\pi^2\,z_0{}^2}\sum_\mu\frac{P_\mu}{\mu^2}\left\{\left(1+\psi\frac{V}{L}\right)\frac{I_0(\mu\,\pi\,r/L)}{I_1(\mu\pi/L)} - \left(1-\psi\frac{V}{L}\right)\frac{2L}{\mu\pi}\right\}\sin\frac{\mu\pi}{L}z\,. \qquad (48)$$

I_0 and I_1 are the modified Bessel functions of order 0 and 1, respectively.

For the function $\Phi_2 = \Phi_2(z)$ one obtains: $\Phi_2(z) = V\,z/L\,.$ (49)

Inserting (36), (48), and (49) into Ohm's law (29), yields the components of the ground state electric current density, including terms of order ψ^2, within the range $0 \leqq r \leqq 1$, $0 \leqq z \leqq L$:

$$j_r^{00} = \frac{\psi(1+\psi\,V/L)}{z_0{}^2}\left\{\frac{r}{2}(z-z_0) - \frac{1}{\pi}\sum_\mu\frac{P_\mu}{\mu}\frac{I_1(\mu\,\pi\,r/L)}{I_1(\mu\pi/L)}\sin\frac{\mu\,\pi}{L}z\right\}, \qquad (50)$$

$$j_\theta^{00} = \frac{r}{2z_0}\left(1+\psi\frac{V}{L}\right) + \frac{\psi^2}{2\pi z_0^3}\, r \sum_\mu \frac{P_\mu}{\mu}\left\{\frac{I_0(\mu\pi r/L)}{I_1(\mu\pi/L)} - \frac{2L}{\mu\pi}\right\}\cos\frac{\mu\pi}{L}z + \frac{\psi^2}{\pi z_0^3}(z-z_0)\sum_\mu \frac{P_\mu}{\mu}\frac{I_1(\mu\pi r/L)}{I_1(\mu\pi/L)}\sin\frac{\mu\pi}{L}z\,, \tag{51}$$

$$j_z^{00} = -\frac{V}{L} + \frac{\psi}{4z_0^2}r^2 - \frac{\psi^2(V/L)}{z_0^2}(z_0-z)^2 - \frac{\psi}{\pi z_0^2}\sum_\mu \frac{P_\mu}{\mu}\left\{\left(1+\psi\frac{V}{L}\right)\frac{I_0(\mu\pi r/L)}{I_1(\mu\pi/L)} - \left(1-\psi\frac{V}{L}\right)\frac{2L}{\mu\pi}\right\}\cos\frac{\mu\pi}{L}z. \tag{52}$$

For vanishing Hall effect ($\psi = 0$), the ground state electric current density is given by

$$\boldsymbol{j}^{00} = \{0, r/2z_0, -V/L\}. \tag{53}$$

According to (29), the azimuthal component hereof stems from the θ-component of the vector product $\boldsymbol{v}^{00}\times\boldsymbol{B}^{00}$, and vanishes if $B_r^{00} = 0$. In this latter case ($z_0 = \infty$), only the imposed homogeneous axial current $-V/L$ prevails, even for a finite Hall parameter.

Using (52), the total electric current for the ground state is found to be

$$J^{00} = 2\pi\int_0^1 j_z^{00}\, r\,\mathrm{d}r = -\pi\left(\frac{V}{L} - \frac{\psi}{8z_0^2}\right). \tag{54}$$

The additional current $\pi\psi/8z_0^2$ — due to the Hall effect and independent of the potential difference V — has its origin in the term $-\psi(\boldsymbol{v}^{00}\times\boldsymbol{B}^{00})\times\boldsymbol{B}^{00}$ of Ohm's law (29), so that its flow direction is given by the vector $\boldsymbol{v}^{00}$. This additional current vanishes for $\psi = 0$, but also for finite values of ψ if the imposed magnetic field degenerates into the purely axial and homogeneous one described by $z_0 = \infty$.

4.3. Solution of the Equations of Motion

Having determined $\boldsymbol{j}^{00}$, we now proceed to solve the coupled system consisting of the linearized first-order equations of continuity (27) and momentum (28) which in component notation read:

$$v_{r|r}^{010} + (1/r)\,v_r^{010} + v_{z|z}^{010} + \varrho_{|z}^{010} = 0\,, \tag{55}$$

$$v_{r|z}^{010} + (1/M^2)\,\varrho_{|r}^{010} = j_\theta^{00}\,B_z^{00}\,, \tag{56}$$

$$v_{\theta|z}^{010} = j_z^{00}\,B_r^{00} - j_r^{00}\,B_z^{00}\,, \tag{57}$$

$$v_{z|z}^{010} + (1/M^2)\,\varrho_{|z}^{010} = -\,j_\theta^{00}\,B_r^{00}\,. \tag{58}$$

The differential Eq. (57) for the azimuthal component v_θ^{010} is integrated directly, taking into account (19), (50), and (52), as well as the initial condition $v_\theta^{010}(r,0) = 0$, to yield:

$$v_\theta^{010}(r,z) = -\frac{V/L}{2z_0}rz + \frac{\psi}{8z_0^3}r^3 z + \frac{\psi}{6z_0^3}r[(z-z_0)^3 + z_0^3] - \frac{\psi(1+\psi V/L)L}{\pi^2 z_0^3}\sum_\mu \frac{P_\mu}{\mu^2 I_1(\mu\pi/L)}\left\{\frac{r}{2}I_0\left(\frac{\mu\pi}{L}r\right)\sin\frac{\mu\pi}{L}z + I_1\left(\frac{\mu\pi}{L}r\right)\cdot\left[\frac{L}{\mu\pi}\sin\frac{\mu\pi}{L}z + (z_0-z)\cos\frac{\mu\pi}{L}z - z_0\right]\right\} + \frac{\psi(1-\psi V/L)L^2}{\pi^3 z_0^3}\, r\sum_\mu \frac{P_\mu}{\mu^3}\sin\frac{\mu\pi}{L}z\,. \tag{59}$$

Because of its derivation, the first-order azimuthal velocity perturbation, v_θ^{010}, is the same for free jet and channel flow. Unlike the other perturbations — to be determined later on — it does not depend on the free stream Mach number M, a result already pointed out by HASIMOTO[7,11]. v_θ^{010} vanishes on the axis $r = 0$, in keeping with obvious symmetry requirements. For vanishing Hall effect ($\psi = 0$), a finite potential difference V causes the plasma to rotate. If on the other hand $V = 0$, plasma rotation is maintained as long as the Hall parameter ψ differs from zero. Plasma rotation will cease only in case the imposed magnetic induction degenerates into a homogeneous, axial field.

To determine the remaining perturbations v_r^{010}, v_z^{010}, and ϱ^{010} the system of first-order Eqs. (55), (56), (58) may be uncoupled to yield a single second-order differential equation for anyone of these three unknowns. If for that purpose we choose the density perturbation, ϱ^{010}, for free jet flow, and the radial component of the perturbation velocity, v_r^{010}, for channel flow the corresponding boundary conditions will have a particularly simple form.

4.3.1. Free jet flow

Integrating Eq. (58) subject to the conditions

$$v_z^{010}(r,0) = 0\,, \tag{60}$$

$$\varrho^{010}(r,0) = 0\,, \tag{61}$$

one obtains:

$$v_z^{010}(r,z) = -(1/M^2)\,\varrho^{010}(r,z) - \int_0^z j_\theta^{00}\,B_r^{00}\,\mathrm{d}\zeta\,. \tag{62}$$

Substituting (62) into (55) gives:

$$v_{r|r}^{010} + \frac{1}{r}\,v_r^{010} + \varrho_{|z}^{010}\left(1 - \frac{1}{M^2}\right) = j_\theta^{00}\,B_r^{00}\,. \tag{63}$$

Differentiating Eq. (63) with respect to z and Eq. (56) with respect to r, and then subtracting yields, because of $\nabla \times \boldsymbol{B}^{00} = 0$:

$$\varrho_{|rr}^{010} + \frac{1}{r}\,\varrho_{|r}^{010} - (M^2 - 1)\,\varrho_{|zz}^{010} = M^2\left\{\frac{1}{r}\,j_\theta^{00}\,B_z^{00} + j_{\theta|r}^{00}\,B_z^{00} - j_{\theta|z}^{00}\,B_r^{00}\right\}. \tag{64}$$

This linear second-order partial differential equation is hyperbolic, elliptic, or parabolic, depending on whether the difference $M-1$ is positive, negative, or equal to zero, respectively.

Integrating Eq. (56) subject to condition

$$v_r^{010}(r,0) = 0\,, \tag{65}$$

we obtain for the radial component of the perturbation velocity:

$$v_r^{010}(r,z) = \int_0^z [j_\theta^{00}\,B_\zeta^{00} - (1/M^2)\,\varrho_{|r}^{010}]\,\mathrm{d}\zeta\,. \tag{66}$$

This completes, in principle, the determination of perturbation velocity and density.

Next to (66), we find the deformed jet boundary, $r_w(z)$. Within our linear approximation, we obtain using relations (24)—(26):

$$\left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)_w = \left(\frac{v_r}{v_z}\right)_{r=1} = \left(\frac{R_m S\, v_r{}^{010}}{1 + R_m S\, v_z^{010}}\right)_{r=1} = R_\mathrm{m} S\,(v_r^{010})_{r=1}\,.$$

Because of $r_w(0) = 1$, this gives:

$$r_w(z) = 1 + R_\mathrm{m} S \int_0^z v_r^{010}(1,\zeta)\,\mathrm{d}\zeta\,. \tag{67}$$

On account of assumptions (20), (22), and (23), the azimuthal component of the ground state electric current density reduces to

$$j_\theta^{00} = (r/2\,z_0)\,(1 + \psi\, V/L)\,. \tag{68}$$

Using (19), the right-hand side of Eq. (64) — abbreviated by Q/r — then reads:

$$\frac{Q}{r} = M^2\,\frac{1 + \psi\, V/L}{z_0{}^2}\,(z_0 - z)\,. \tag{69}$$

In addition to (61), the following boundary condition holds:

$$\varrho^{010}(1,z) = 0\,, \tag{70}$$

which because of our linear approximation refers again to the undeformed cylindrical jet boundary where the plasma pressure is assumed to equal the ambient gas pressure.

The solution of differential Eq. (64) will be represented by superposition of its eigenfunctions, namely the Bessel functions of zeroth order $J_0(\lambda_\mu r)$, with the equation

$$J_0(\lambda_\mu) = 0\,, \qquad \mu = 1, 2, \ldots \tag{71}$$

defining the non-zero eigenvalues.

Expanding Q/r in terms of these eigenfunctions, one obtains[10]:

$$\frac{Q}{r} = 2\,M^2\,\frac{1 + \psi\, V/L}{z_0{}^2}\,(z_0 - z)\sum_\nu \frac{J_0(\lambda_\nu\, r)}{\lambda_\nu J_1(\lambda_\nu)}\,, \tag{72}$$

$J_1(\lambda_\nu)$ being the first-order Bessel function for the eigenvalues λ_ν as defined by Eq. (71).

Assuming a solution $\varrho^{010}(r,z)$ exists, we now determine its Fourier coefficients

$$\chi_\mu(z) = \frac{2}{[J_1(\lambda_\mu)]^2}\int_0^1 \varrho^{010}(r,z)\,r\,J_0(\lambda_\mu\, r)\,\mathrm{d}r \tag{73}$$

in terms of the orthogonal and complete system of functions $J_0(\lambda_\mu r)$, $\mu = 1, 2, \ldots$. To this end, we multiply Eq. (64) by $J_0(\lambda_\mu r)$, and integrate termwise over the fundamental domain. Using relations (70)—(73), this gives the following inhomogeneous differential equation for the Fourier coefficients:

$$\chi_\mu''(z) + \frac{\lambda_\mu{}^2}{M^2 - 1}\,\chi_\mu(z) = -\frac{2\,M^2}{M^2 - 1}\,\frac{1 + \psi\, V/L}{z_0^2}\,\frac{1}{\lambda_\mu\, J_1(\lambda_\mu)}\,(z_0 - z)\,. \tag{74}$$

The character of function $\chi_\mu(z)$ is determined by the positive or negative value of the denominator $M^2 - 1$, characterizing supersonic or subsonic flow, respectively. Since no boundary condition at $z = L$ is available, Eq. (64) will not admit, in general, convergent solutions in the elliptic case ($M < 1$). Also, in the parabolic case ($M = 1$), the linearized problem has no physically meaningful solutions. Starting from (74) and (73), and taking into account condition (61), in the hyperbolic case ($M > 1$) we obtain the density perturbation

$$\varrho^{010}(r,z) = \sum_{\mu} \left\{ c_\mu \sin \frac{\lambda_\mu}{\delta} z - \frac{2\,M^2}{\lambda_\mu{}^3 J_1(\lambda_\mu)} \frac{1+\psi\,V/L}{z_0{}^2} \left[z_0 - z + \frac{\delta}{\lambda_\mu} \sin \frac{\lambda_\mu}{\delta} z - z_0 \cos \frac{\lambda_\mu}{\delta} z \right] \right\} J_0(\lambda_\mu r)\,, \quad (75)$$

wherein $\delta \equiv \sqrt{M^2 - 1}$.

The constants of integration c_μ can only be calculated after having determined the remaining perturbations.

Substituting (19), (68), and (75) into (66) and (62), respectively, we obtain the radial and axial components of the perturbation velocity as follows:

$$v_r^{010}(r,z) = \frac{1+\psi\,V/L}{4z_0{}^2} r z (2z_0 - z) + \sum_{\mu} \left\{ \frac{2\delta}{M^2} c_\mu \sin^2 \frac{\lambda_\mu}{2\delta} z \right.$$
$$\left. - \frac{2(1+\psi\,V/L)}{z_0{}^2} \frac{1}{\lambda_\mu{}^2 J_1(\lambda_\mu)} \left[z_0 z - \frac{z^2}{2} - \frac{z_0\,\delta}{\lambda_\mu} \sin \frac{\lambda_\mu}{\delta} z + \frac{2\delta^2}{\lambda_\mu{}^2} \sin^2 \frac{\lambda_\mu}{2\delta} z \right] \right\} \times J_1(\lambda_\mu r)\,, \quad (76)$$

$$v_z^{010}(r,z) = -\frac{1}{M^2} \varrho^{010}(r,z) - \frac{1+\psi\,V/L}{4z_0{}^2} r^2 z\,. \quad (77)$$

Eqs. (56) and (58) are satisfied by solutions (75) to (77) for *arbitrary* values of the constants c_μ. These latter are determined, therefore, by substituting (75)–(77) into Eq. (55) which — because of condition (65) — reads at $z = 0$:

$$v_{z|z}^{010}(r,0) + \varrho_{|z}^{010}(r,0) = 0\,. \quad (78)$$

Substituting (77) into (78) yields the following second initial condition for the density perturbation:

$$\varrho_{|z}^{010}(r,0) = \frac{M^2(1+\psi\,V/L)}{4(M^2-1)\,z_0{}^2} r^2\,. \quad (79)$$

Using (75) and (79), the constants of integration are:

$$c_\mu = \frac{M^2(1+\psi\,V/L)}{2\delta\,z_0{}^2} \frac{1-(4/\lambda_\mu{}^2)}{\lambda_\mu{}^2 J_1(\lambda_\mu)}\,, \quad \mu = 1,2,\ldots\,. \quad (80)$$

4.3.2. Channel flow

The above conditions (60), (61), and (65) remain valid here. Instead of (70), though, we have to consider now the boundary conditions

$$v_r^{010}(0,z) = 0\,, \quad v_r^{010}(1,z) = 0\,. \quad (81)$$

Solving Eq. (62) for ϱ^{010}, gives:

$$\varrho^{010}(r,z) = -M^2 v_z^{010}(r,z) - M^2 \int_0^z j_\theta^{00} B_r^{00}\,\mathrm{d}\zeta\,. \quad (82)$$

Substituting (82) into Eqs. (55) and (56), respectively, yields:

$$v_{r|r}^{010} + \frac{1}{r} v_r^{010} - (M^2-1)\,v_{z|z}^{010} = M^2 j_\theta^{00} B_r^{00}\,, \quad (83)$$

$$v_{r|z}^{010} - v_{z|r}^{010} = j_\theta^{00} B_z^{00} + \frac{\delta}{\delta r} \int_0^z j_\theta^{00} B_r^{00}\,\mathrm{d}\zeta\,. \quad (84)$$

Differentiating Eq. (83) with respect to r and Eq. (84) multiplied by $1 - M^2$ with respect to z, and adding, gives[12]:

$$\begin{aligned} v_{r|rr}^{010} + (1/r)\,v_{r|r}^{010} - (1/r^2)\,v_r^{010} - (M^2-1)\,v_{r|zz}^{010} &= j_{\theta|r}^{00} B_r^{00} + j_\theta^{00} B_{r|r}^{00} \\ &\quad - (M^2-1)\,(j_{\theta|z}^{00} B_z^{00} + j_\theta^{00} B_{z|z}^{00})\,. \end{aligned} \quad (85)$$

Eq. (85) is hyperbolic, elliptic, or parabolic, depending on whether the difference $M - 1$ is positive, negative, or equal to zero, respectively.

By analogy to the procedure in Sect. 4.3.1, the right-hand side of Eq. (85) — abbreviated by P/r — reduces to

$$\frac{P}{r} = \frac{M^2(1+\psi\,V/L)}{2z_0{}^2} r\,. \quad (86)$$

Differential Eq. (85) is solved by superposition of its eigenfunctions which are the Bessel functions of first order $J_1(\lambda_\nu r)$, where the equation

$$J_1(\lambda_\nu) = 0\,, \quad \nu = 1,2,\ldots \quad (87)$$

defines the non-zero eigenvalues.

Expanding P/r in terms of these eigenfunctions, gives:

$$\frac{P}{r} = -\frac{M^2(1+\psi\,V/L)}{z_0{}^2} \sum_{\nu} \frac{J_1(\lambda_\nu r)}{\lambda_\nu J_0(\lambda_\nu)}\,. \quad (88)$$

Assuming the existence of a solution $v_r^{010}(r,z)$, we have to determine its Fourier coefficients

$$\psi_\nu(z) = \frac{2}{[J_0(\lambda_\nu)]^2} \int_0^1 v_r^{010}(r,z)\,r\,J_1(\lambda_\nu r)\,\mathrm{d}r \quad (89)$$

in terms of the orthogonal and complete system of functions $J_1(\lambda_\nu r)$, $\nu = 1, 2, \ldots$. By making use of the orthogonality relations for Bessel functions as well as of (81) and (87), we find:

$$\psi_\nu''(z) + \frac{\lambda_\nu^2}{M^2-1}\,\psi_\nu(z) = \frac{M^2}{M^2-1}\,\frac{1+\psi V/L}{z_0^2}\,\frac{1}{\lambda_\nu J_0(\lambda_\nu)}\,. \tag{90}$$

Again, as in Sect. 4.3.1, we only have to consider the supersonic case ($M > 1$). Using the corresponding solution of differential Eq. (90), and definition (89), as well as the initial condition (65), the following representation is obtained for the radial component of the perturbation velocity:

$$v_r^{010}(r,z) = \frac{2M^2(1+\psi\, V/L)}{z_0^2}\sum_\nu \frac{J_1(\lambda_\nu r)}{\lambda_\nu^3 J_0(\lambda_\nu)} \cdot \left\{\sin^2\frac{\lambda_\nu}{2\delta}z + C_\nu \sin\frac{\lambda_\nu}{\delta}z\right\}. \tag{91}$$

Substituting (19), (68), (91) into Eq. (83), and integrating gives [12]:

$$v_z^{010}(r,z) = \frac{M^2(1+\psi\, V/L)}{\delta^2 z_0^2}\left\{-\frac{r^2 z}{4} + 2\sum_\nu \frac{J_0(\lambda_\nu r)}{\lambda_\nu^2 J_0(\lambda_\nu)} \cdot \left[\frac{z}{2} - \frac{\delta}{2\lambda_\nu}\sin\frac{\lambda_\nu}{\delta}z + \frac{2\delta}{\lambda_\nu}C_\nu \sin^2\frac{\lambda_\nu}{2\delta}z\right]\right\}. \tag{92}$$

(82) together with (19) and (68) yield finally for the density perturbation:

$$\varrho^{010}(r,z) = -M^2 v_z^{010}(r,z) - \frac{M^2(1+\psi\, V/L)}{4 z_0^2}\, r^2 z\,. \tag{93}$$

For determining the constants C_ν, we refer to Eq. (84) which, at $z = 0$, and using (19), (68), and (60), supplies the following second initial condition for v_r^{010}:

$$v_{r|z}^{010}(r,0) = \frac{1+\psi\, V/L}{2z_0}\, r\,. \tag{94}$$

(91) substituted into (94), then gives:

$$C_\nu = -(\delta z_0/2M^2)\,\lambda_\nu\,. \qquad \nu = 1,2,\ldots. \tag{95}$$

4.4. *Subsonic Flow*

The initial conditions referring to $z = 0$ used so far, lead to convergent solutions for the perturbations v_r^{010}, v_z^{010}, and ϱ^{010} only in case of supersonic flow. Convergent solutions for the elliptic problem ($M < 1$) are possible only if for the variable in question a boundary condition at $z = L$ is available, instead of the second initial condition at $z = 0$. Of some practical interest might conceivably be the case of subsonic channel flow with the additional boundary condition

$$v_r^{010}(r,L) = 0\,. \tag{96}$$

To uniquely solve the system of Eqs. (55), (56), (58) for this case, one of conditions (60), (61) must be dispensed with, maintaining though condition (65). Dropping (61), one obtains in a similar fashion as in Sect. 4.3.2 the following results:

$$v_r^{010}(r,z) = \frac{2M^2(1+\psi\, V/L)}{z_0^2}\sum_\mu \frac{J_1(\lambda_\mu r)}{\lambda_\mu^3 J_0(\lambda_\mu)}\,\frac{\sinh(\lambda_\mu z/2\gamma)}{\cosh(\lambda_\mu L/2\gamma)}\sinh\frac{\lambda_\mu}{2\gamma}(L-z)\,, \tag{97}$$

$$v_z^{010}(r,z) = \frac{M^2(1+\psi\, V/L)}{\gamma^2 z_0^2}\left\{\frac{r^2 z}{4} - 2\sum_\mu \frac{J_0(\lambda_\mu r)}{\lambda_\mu^2 J_0(\lambda_\mu)}\left[\frac{z}{2} - \frac{\gamma}{\lambda_\mu}\,\frac{\sinh(\lambda_\mu z/2\gamma)}{\cosh(\lambda_\mu L/2\gamma)}\cosh\frac{\lambda_\mu}{2\gamma}(L-z)\right]\right\}, \tag{98}$$

$$\varrho^{010}(r,z) = -M^2 v_z^{010}(r,z) + \frac{M^2(1+\psi\, V/L)}{z_0^2}\left\{\frac{r^2(z_0-z)}{4} - \frac{M^2}{\gamma}\sum_\mu \frac{1-J_0(\lambda_\mu r)}{\lambda_\mu^3 J_0(\lambda_\mu)}\tanh\frac{\lambda_\mu L}{2\gamma}\right\}, \tag{99}$$

wherein $\gamma \equiv \sqrt{1-M^2}$, and λ_μ are the zeros of Bessel function J_1. At $z = 0$, (99) yields the following initial distribution of the density perturbation, differing from (61):

$$\varrho^{010}(r,0) = \frac{M^2(1+\psi\, V/L)}{z_0^2}\left\{\frac{r^2 z_0}{4} - \frac{M^2}{\gamma}\sum_\mu \frac{1-J_0(\lambda_\mu r)}{\lambda_\mu^3 J_0(\lambda_\mu)}\tanh\frac{\lambda_\mu L}{2\gamma}\right\}. \tag{100}$$

5. Discussion of the Results

Plasma velocity and density for both free jet and channel flow are obtained by substituting (25), (26), and the perturbations as calculated in Sect. 4.3 to 4.4 into (24), according to the following scheme:

$$\boldsymbol{v} = \{0,0,1\} + R_\text{m} S\{v_r^{010}, v_\theta^{010}, v_z^{010}\}\,, \tag{101}$$

$$\varrho = 1 + R_\text{m} S\,\varrho^{010}\,. \tag{102}$$

The discussion will be conducted in two stages. To begin with, those conclusions are to be discussed that may be drawn without a numerical evaluation of the closed form solutions obtained in Sect. 4.3 to 4.4. And these conclusions are seen to be valid for both free jet and channel flow. The radial component of the flow velocity, v_r, vanishes on the axis, $r = 0$, as required because of the axial symmetry of the problem. For a homogeneous, axially directed

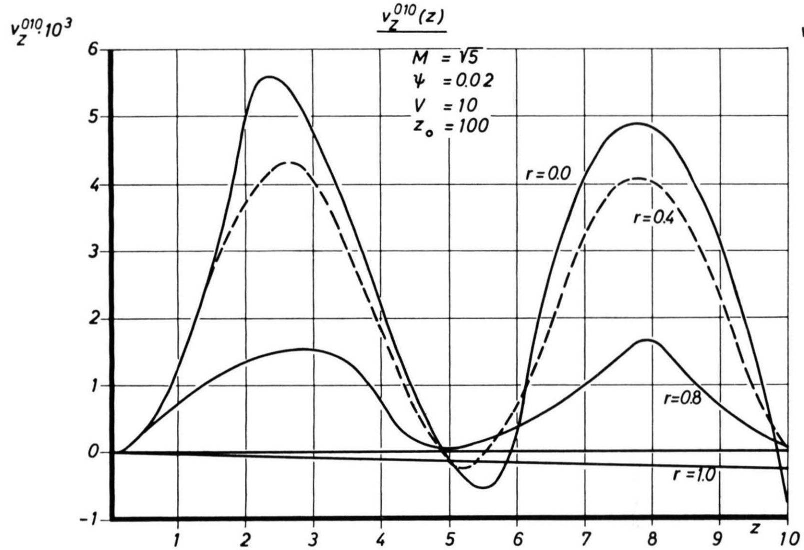

Fig. 2. Axial component of free jet perturbation velocity in a diverging magnetic field for $M = \sqrt{5}$.

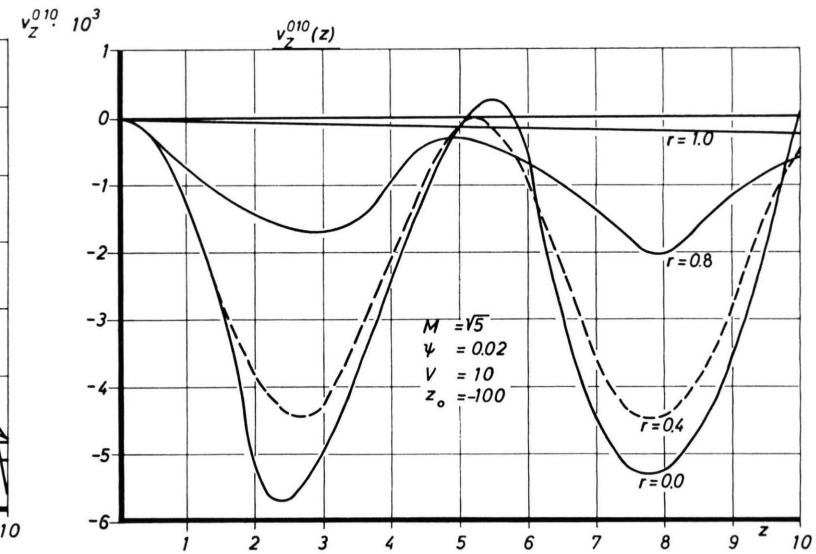

Fig. 4. Axial component of free jet perturbation velocity in a converging magnetic field for $M = \sqrt{5}$.

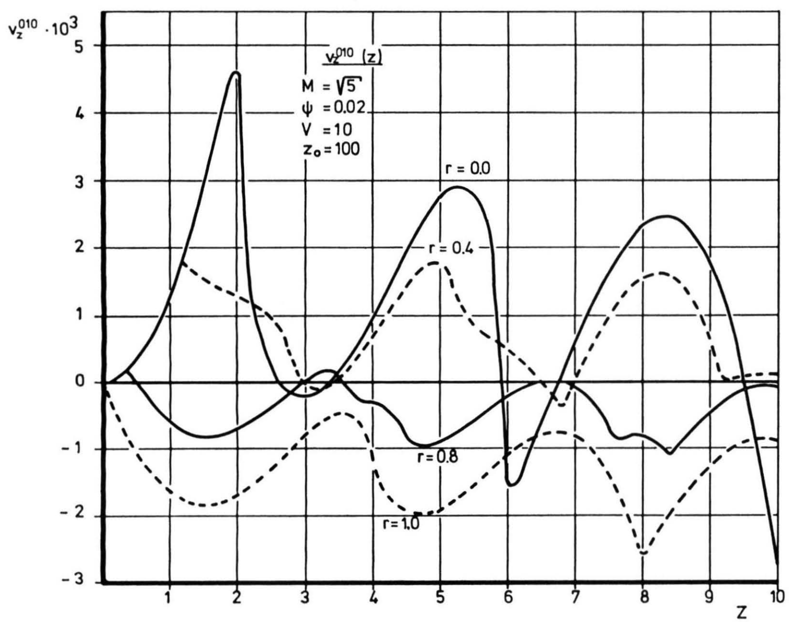

Fig. 3. Axial component of channel flow perturbation velocity in a diverging magnetic field for $M = \sqrt{5}$.

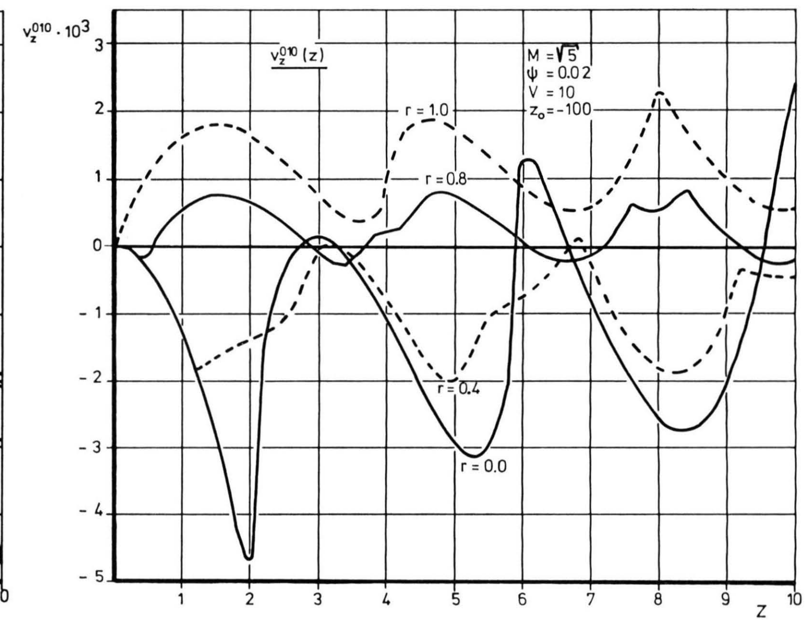

Fig. 5. Axial component of channel flow perturbation velocity in a converging magnetic field for $M = \sqrt{5}$.

imposed magnetic field ($z_0 = \infty$), the perturbations v_r^{010}, v_z^{010}, and ϱ^{010} vanish identically because in this case the azimuthal component of the ground state electric current density, j_θ^{00}, is equal to zero too. Because of using approximation (68) for j_θ^{00} where the Hall parameter ψ and the imposed electric field magnitude V/L appear only within the typical combination $1 + \psi\,(V/L)$, the same holds true, of course, for the perturbations v_r^{010}, v_z^{010}, and ϱ^{010}. Consequently, for vanishing Hall effect ($\psi = 0$), these perturbations do not depend — even for finite values of z_0 — on the potential difference V, but only on the imposed induction $\boldsymbol{B}^{00}$. On the other hand, the above perturbations may be influenced by Hall parameter variations only if a finite potential difference is maintained between entrance and exit sections.

For the second stage of the discussion, further information is obtained by a numerical evaluation of the previous results. In accordance with our assumptions (20) and (22), the characteristic constants have been chosen as follows:

$$\psi = 0{,}02; \quad V = 10; \quad L = 10;$$

$z_0 = +\,100,\ -\,100$ for the slightly diverging or converging imposed magnetic induction, respectively.

First, we examine the electric current density and the azimuthal component of the perturbation velocity. These results refer likewise to free jet and channel flow, and are valid for both supersonic and subsonic conditions. Everywhere, the numerical values for the radial component of the electric current density, j_r^{00}, are found to be smaller by several orders of magnitude than the remaining variables. For all practical purposes, j_r^{00} may thus be assumed to vanish identically. The Hall effect gives rise to an additional axial component of the electric current density: $j_z^{00} + V/L$ which, in our approximation, is practically independent of the coordinates r and z. The azimuthal component j_θ^{00}, as given by (68), is a linear function of r, and independent of z. The azimuthal component of the perturbation velocity, v_θ^{010}, is practically linear in r and z, vanishing on the axis, $r = 0$, and at the entrance section, $z = 0$. The direction of plasma rotation changes with the sign of V.

The remaining perturbations are different for jet and channel flow, as well as for supersonic and subsonic conditions[10, 12], as illustrated in Figs. 2—12. Figs. 2 and 3 show the local variations of the axial component of the perturbation velocity, v_z^{010}, for a slightly diverging imposed magnetic field in case of free jet and channel flow, respectively. The axial velocity increases in both cases have their absolute maxima — larger for free jet than for channel flow — on the symmetry axis. But, whereas in the former case, v_z^{010} is positive practically everywhere, a considerable deceleration takes place in the layers adjacent to a channel wall ($r = 1{,}0$). Figs. 4 and 5 illustrate the local variations of v_z^{010} under the same conditions as before, but for a slightly converging magnetic induction field. With the exception of the free jet boundary region where an insignificant deceleration persists in any case, the v_z^{010}-curves for $z_0 = -\,100$ are obtained approximately from those for $z_0 = +\,100$ by means of reflexion with respect to the abscissa.

Fig. 6 is to show the contour deformations for the free jet case just considered ($\Delta r \equiv r_w - 1$). The jet expands in the divergent $\boldsymbol{B}^{00}$-field and contracts if the applied field converges ("magnetic nozzle").

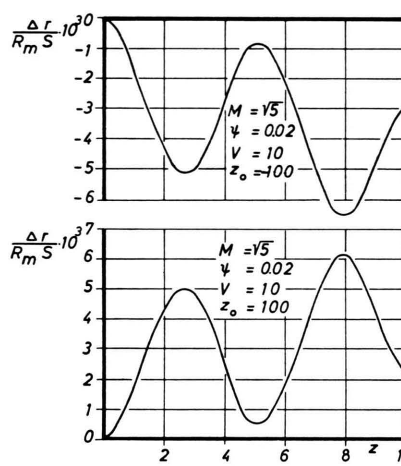

Fig. 6. Free jet contour deformations in diverging and converging magnetic fields for $M = \sqrt{5}$.

According to (77) and (93), the density perturbation for both free jet and channel flow in the supersonic case is given by the same expression

$$\varrho^{010}(r, z) = -\,M^2\,v_z^{010}(r, z) - M^2\,\frac{1 + \psi\,V/L}{4 z_0^2}\,r^2 z\,. \tag{103}$$

In general, we may neglect herein — on account of assumptions (20) and (22) — the second term on the right-hand side of (103) relative to the first one. Therefore, the density perturbation may be obtained in good approximation by reflecting the v_z^{010}-curves in the abscissa and stretching them by a scale factor M^2. This is illustrated by comparing Figs. 3 and 7.

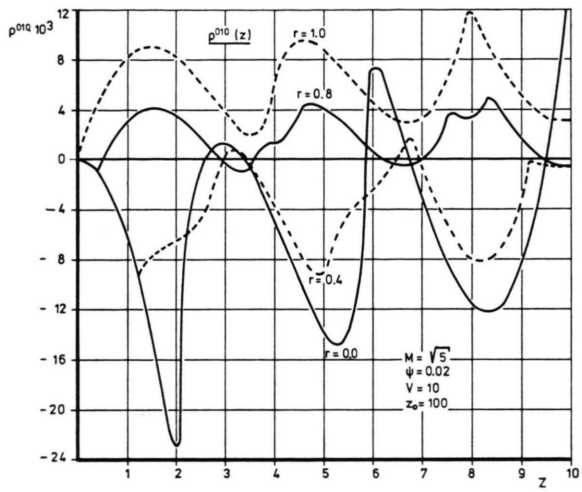

Fig. 7. Channel flow density perturbation in a diverging magnetic field for $M = \sqrt{5}$.

The perturbations v_r^{010}, v_z^{010}, and ϱ^{010} are nearly periodical in z-direction. An approximate value for the corresponding "wave length" Λ may be calculated by ignoring all but the first-order terms in the series expansions in (75), (76), and (91), (92), respectively. This yields:

$$\Lambda = \frac{2\pi\sqrt{M^2 - 1}}{\lambda}, \tag{104}$$

wherein $\lambda = 2{,}4048$ for free jet flow, and
$\lambda = 3{,}8313$ for channel flow

are the first zeros of the Bessel functions J_0 and J_1, respectively. Consequently, the "waviness" of the perturbations and of the free jet contour will disappear over the entire length L if the free stream Mach number M satisfies the inequality

$$M > \sqrt{(\lambda^2 L^2/\pi^2) + 1}\,. \tag{105}$$

For the free jet case — with $M = 8$ —, this is illustrated in Fig. 8 for v_z^{010}, and in Fig. 9 for the expanding and contracting jet contours, all curves being essentially monotonous over the entire length L. Fig. 10, representing $v_z^{010}(z)$ for different values of M, also shows clearly the increase of Λ with the free stream Mach number, in case of channel flow under the influence of a diverging $\boldsymbol{B}^{00}$-field.

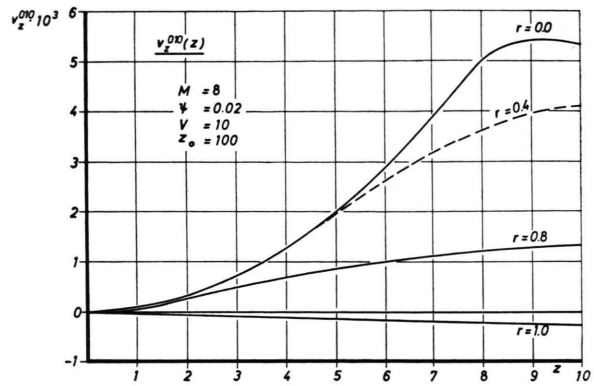

Fig. 8. Axial component of free jet perturbation velocity in a diverging magnetic field for $M = 8$.

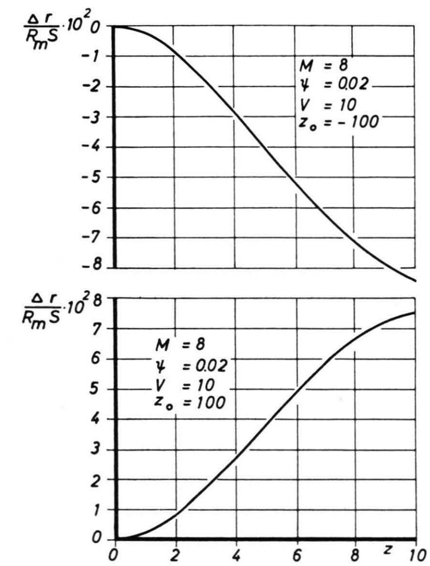

Fig. 9. Free jet contour deformations in diverging and converging magnetic fields for $M = 8$.

In Fig. 11, we again consider channel flow, illustrating the local variations of v_z^{010} for $M = 13$, the imposed magnetic induction now being slightly *converging* in z-direction ($z_0 = -100$). It is therefore the outer layers of the plasma which are being accelerated here, the inner ones being slowed down. This effect might possibly be utilized in the case of viscous flows of ionized gases to partially compensate for boundary layer effects. In Fig. 12 we compare the effectiveness of plasma acceleration by means of an applied slightly divergent magnetic field for free jet and channel flow. This is done by

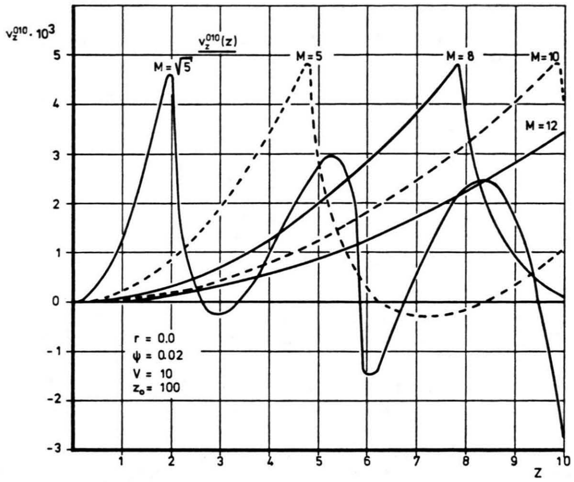

Fig. 10. Channel flow axial velocity perturbation on the axis for a diverging magnetic field.

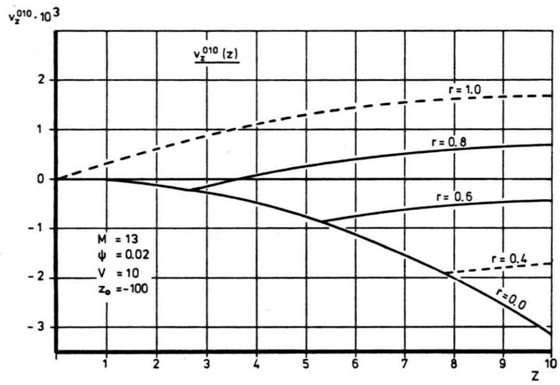

Fig. 11. Axial component of channel flow perturbation velocity in a converging magnetic field for $M = 13$.

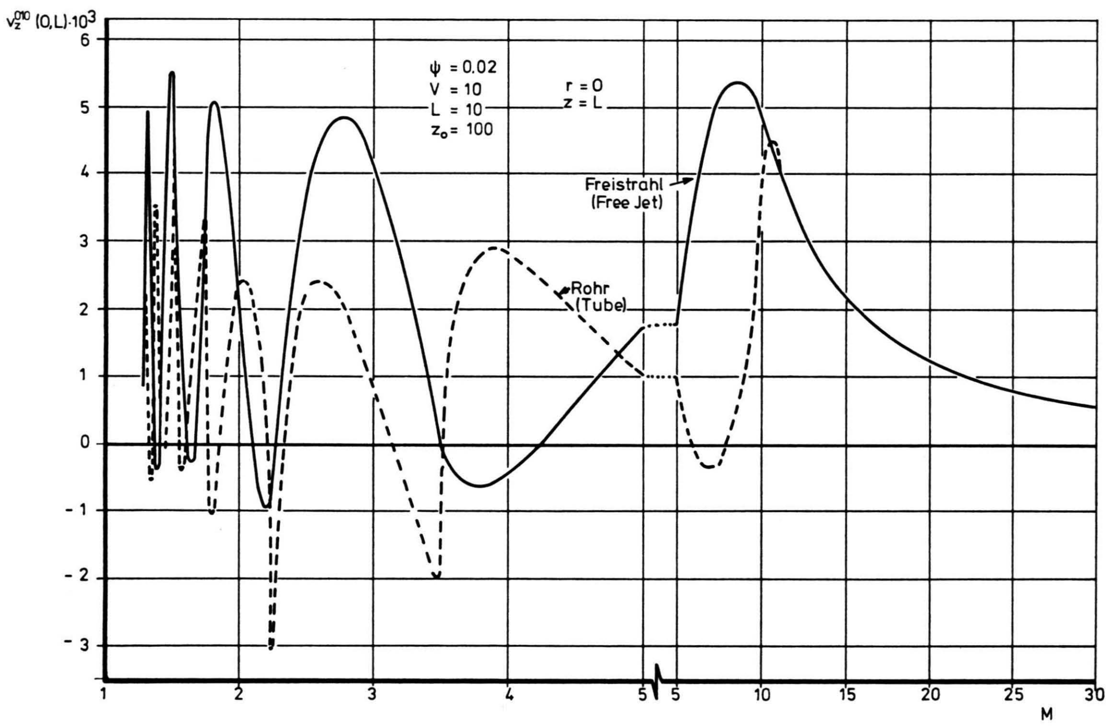

Fig. 12. Mach number dependence of free jet and channel flow axial velocity perturbations for a diverging magnetic field.

plotting the values of v_z^{010} taken at the center of the exit section ($r = 0$, $z = L$) as a function of the free stream Mach number M. Higher acceleration rates result for free jet flow. This is explained by the fact that the cross section available for the supersonic free jet is — due to its expansion in the diverging $\boldsymbol{B}^{00}$-field — larger than for constant-area channel flow.

It may be shown that the free jet problem admits no physically meaningful solutions for the subsonic case, unless a constraint on $\varrho^{010}(r, L)$ is formulated which seems impossible, though, from a physical point of view. On the other hand, in Sect. 4.4 we found a *subsonic* solution for *channel flow* by adopting boundary condition (96). But this latter constraint yielded the rather unrealistic density perturbation (100) at $z = 0$. According to (98), v_z^{010} depends only on the square of the characteristic induction constant z_0. Therefore, the axial velocity distribution is the same for both diverging and converging $\boldsymbol{B}^{00}$-fields. A numerical evaluation of (98) shows that v_z^{010} is an everywhere positive, monotonously increasing function of z for all values of r. ϱ^{010} is — according to (99) — a monotonously decreasing function of z for every r, the initial values at $z = 0$ being given by (100).

Acknowledgments

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